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A Note on the Rate of Convergence of Best Weighted Approximation

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1. INTRODUCTION

Let n be a fixed positive integer and let $x_1, ..., x_n$ be fixed, distinct elements of $[a, b]$. Let $\beta_1, ..., \beta_n$ be corresponding fixed nonnegative real numbers. For each k let $m_k = [\beta_k]$ be the greatest integer $\leqslant \beta_k$ and let $\alpha_k = \beta_k - m_k$ so that $0 \leqslant \alpha_k < 1$, $k = 1, ..., n$. Let $w(x) = [\prod_{i=1}^{n} | x - x_i |^{\beta_i}]^{-1}$, and $E = [a, b] - \{x_1, ..., x_n\}$.

We say that a function f is approximable with respect to the weight w on E if for any $\epsilon > 0$ there is a polynomial p such that $w(x)| f(x) - p(x)| < \epsilon$ for each x in E.

In [1] the following theorem is proved:

THEOREM A. *f is approximable with respect to w if and only if*

$$f(x) = h(x) \prod_{k=1}^{n} (x - x_k)^{m_k} | x - x_k |^{\alpha_k} + p(x),$$

where $h \in C[a, b]$; $h(x_k) = 0$ if $\alpha_k > 0$; and p is a polynomial which may be assumed to have degree $m_1 + \cdots + m_n + n - 1$ or less.

Define $E_m(f)$ to be the best uniform approximation to $f \in C[a, b]$ by polynomials of degree m or less. Similarly, for w as above define

$$E_{m,w}(f) = \inf_{p \in H_m} \max_{x \in E} w(x)| f(x) - p(x)|.$$

(H_m is the set of polynomials of degree m or less).

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If f is approximable with respect to w, then by Theorem A f has the form

$$f(x) = h(x) \prod_{k=1}^{n} (x - x_k)^{m_k} | x - x_k |^{\alpha_k} + p(x),$$

where h and p are as in the theorem. In this paper we compare $E_{m,w}(f)$ with ordinary best approximation to h for certain w.

2. The Main Theorem

Theorem. *If $\beta_1, ..., \beta_n$ are integers and if f is approximable with respect to w, then*

$$E_{m,w}(f) = E_{m-M}(h) \qquad \textit{for} \quad m \geqslant M + n.$$

(*Here $M = \beta_1 + \beta_2 + \cdots + \beta_n$ and h is as in Theorem A.*)

Proof. Let Q_{m-M} be the polynomial of degree $m - M$ of best uniform approximation to h on $[a, b]$, and define

$$P_m(x) = \prod_{i=1}^{n} (x - x_i)^{\beta_i} Q_{m-M}(x) + p(x).$$

Then the degree of P_m is at most m since the first term is of degree m and the degree of p can be chosen to be $M + n - 1$ (see [1]).

Thus we have for each x in E

$$w(x)| f(x) - P_m(x)| = | h(x) - Q_{m-M}(x)| \leqslant E_{m-M}(h).$$

That is,

$$E_{m,w}(f) \leqslant E_{m-M}(h). \tag{1}$$

We now prove the inequality is in fact equality. With this in mind we suppose that $E_{m,w}(f) < E_{m-M}(h)$. Let $P_m{}^*$ be a polynomial of degree less than or equal to m for which

$$\max_{x \in E} | w(x)|| f(x) - P_m{}^*(x)| = E_{m,w}(f).$$

It is easy to show that $P_m{}^*$ exists and may be written in the form

$$P_m{}^*(x) = p(x) + Q(x) \prod_{i=1}^{n} (x - x_i)^{\beta_i},$$

where Q is a polynomial of degree $m - M$, and p is as in Theorem A.

Thus we have for each x in E,

$$\begin{aligned} |h(x) - Q(x)| &= w(x) \left| h(x) \prod_{i=1}^{n} (x - x_i)^{\beta_i} - Q(x) \prod_{i=1}^{n} (x - x_i)^{\beta_i} \right| \\ &= w(x) |f(x) - P_m^*(x)| \\ &\leqslant E_{m,w}(f) < E_{m-M}(h). \end{aligned}$$

This contradiction together with inequality (1) establish the desired result.

Remarks. It is of course difficult in general to determine the continuity and smoothness properties of h which would allow the use of the well known estimates on the degree of convergence. In particular cases, however, it is relatively easy to find explicit expressions for h and thus obtain an estimate for $E_{m,w}(f)$.

Reference

1. J. A. Roulier, A note on weighted approximation on a closed interval, *An. Acad. Brasil. Ci.* **43** (1971), 347–351.

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